Gases in two dimensions: universal thermodynamics and its consequences

Dragoş-Victor Anghel

Department of Physics, University of Oslo, PO Box 1048 Blindern, N-0316 Oslo, Norway
and
NIPNE—'HH', PO Box MG-6, RO-76900 Bucureşti, Măgurele, Romania

E-mail: dragos@fys.uio.no

Abstract
I discuss ideal and interacting quantum gases obeying general fractional exclusion statistics. For systems with constant density of single-particle states, described in the mean field approximation, the entropy depends neither on the microscopic exclusion statistics, nor on the interaction. Such systems are called *thermodynamically equivalent* and I show that the microscopic reason for this equivalence is a one-to-one correspondence between the excited states of these systems. This provides a method, different from the bosonization technique, of transforming between systems of different exclusion statistics. In the last section the macroscopic aspects of this method are discussed.

In appendix A, I calculate the fluctuation of the ground-state population of a condensed Bose gas in a grand-canonical ensemble and mean field approximation, while in appendix B I show a situation where although the system exhibits fractional exclusion properties on microscopic energy intervals, a rigorous calculation of the population of single-particle states reveals a condensation phenomenon. This also implies a malfunction of the usual and simplified calculation technique of the most probable statistical distributions.

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1. Introduction

Considerable interest has been shown in recent years in the study of particle systems that exhibit fractional exclusion statistics (FES)—a model introduced by Haldane in [1] and which applies, among others, to quasiparticle excitations at the lowest Landau level in the fractional quantum Hall effect and spinon excitations in a spin-$\frac{1}{2}$ quantum antiferromagnet [1]. The general thermodynamic properties of these systems have been deduced mainly by Isakov [2, 3] and Wu [4]. Isakov also showed that anyons [5] and one-dimensional (1D)

systems of particles described by the thermodynamic Bethe ansatz (TBA) [6–8] have the same thermodynamic behaviour as the systems exhibiting FES. Since a similar thermodynamic behaviour of two systems implies a similarity between the *excitation* spectra and eventually—depending on the ensemble in which the thermodynamics is discussed—the same ground-state energies as functions of the particle numbers, this is irrelevant for comparing the microscopic properties of the constituent particles. Sutherland and Iguchi generalized the concept of the Bethe ansatz from one dimension to two and three dimensions (2D, 3D), showing that bosons and fermions described by this model also exhibit fractional exclusion statistics at the microscopic level [9].

The Bethe ansatz was first used to calculate Hamiltonian eigenvectors of 1D spin chains with anisotropic spin–spin interaction (see, for example, [10], and references therein) and then applied, starting with Lieb and Liniger [6], to interacting trapped particles. The model applies in general to dilute systems, with short-range two-body interaction. At the other extreme are dense systems with long range interaction, as compared to interparticle distance, which may be described in the mean field approximation (MFA). Nevertheless, MFA is also applied to dilute systems.

Murthy and Shankar were the first to modify the MFA model for a Fermi system [11]—let me call this new model MFA$'$—by redistributing the mean field interaction among single-particle energies, so that a particle of energy ϵ is assumed to interact just with particles of energy $\epsilon' < \epsilon$ (see appendix B). Within the MFA$'$ model, the interacting Fermi system is equivalent to an ideal Haldane system of statistics parameter α which depends on the interaction strength. Independently, Hansson *et al* [12] used the MFA$'$ model to define single-particle energies in a field theory of anyons at the lowest Landau level, while Isakov and Viefers [13], among other things, showed again that this model reproduces Haldane's FES. All these results have been obtained for constant DOS in the ideal systems.

More recently, Bhaduri *et al* [14] also showed that a repulsively interacting 2D MFA Bose gas has identical thermodynamic parameters to an ideal FES gas with α again fixed by the interaction strength. Continuing [14], Hansson *et al* [15] discussed the applicability of the MFA to bosons with repulsive delta function interaction, in quasi 2D traps, and again transformed the MFA into MFA$'$ to calculate the thermodynamic parameters of the corresponding FES gas.

In what follows I shall say that two systems are *thermodynamically equivalent* if they have the same entropy as a function of temperature, at fixed volume (or external potential) and particle number. The amazing thermodynamic similarity in 2D between interacting Bose or Fermi systems, and ideal FES systems with properly chosen statistics parameter α rests on their thermodynamic equivalence.

To show the special character of nonrelativistic 2D ideal systems, I will start by deriving unified expressions for their thermodynamic quantities in terms of polylogarithmic functions [16]. This bridges Lee's description of bosons and fermions [17, 18] through the intermediate statistics of haldons and emphasizes in a most simple way the thermodynamic equivalence of the systems with constant DOS of any statistics. The thermodynamic equivalence of 2D Bose and Fermi systems was proved by several authors [19–23]. The fact that the temperature dependence of the thermodynamic potentials of Haldane systems with constant DOS is independent of statistics was observed before [24]. Here I merely identify the functions involved, which makes the writing and manipulation of the thermodynamic quantities much easier and compact. In section 2.2, I give the microscopic explanation of this equivalence, which rests *not on a microscopic similarity between systems, but on a one-to-one correspondence between the excited states of these systems*. This also gives us a transformation method between systems of different statistics, which will be extended to more general spectra elsewhere [25]. In section 3.2, based only on macroscopic arguments,

I show that all the thermodynamic quantities describing 2D systems in the Thomas–Fermi approximation and in any trapping potential, depend only on the temperature and the ground-state energy density, which is a function of particle density. This leads to the definition of classes of thermodynamically equivalent systems. Since the proof is based only on macroscopic arguments, it shows explicitly that similar thermodynamic behaviour does not imply microscopic similarity. Based on sections 2 and 3, I can say that the transformation from MFA to MFA$'$ merely sets an abstract point of view or counting rule. In appendix A, I calculate the fluctuation of the ground-state population of a condensed Bose gas in the MFA approximation and finally, in appendix B, I compare in more detail the results given by MFA and MFA$'$ models. I show that in the case of interacting bosons in 2D boxes, depending on the choice of a parameter, MFA$'$ may lead to condensation on the lowest energy level, in contrast to MFA.

2. Ideal Haldane systems with constant density of states

2.1. Thermodynamic equivalence

In what follows I shall consider Haldane systems with DOS of the form $\sigma(\epsilon) = C\epsilon^s$ ($s > -1$), where C is a constant and ϵ is the single-particle energy. If the system (of nonrelativistic particles) is in a d-dimensional (dD) container with no external fields, C is proportional to the hypervolume and $s = d/2 - 1$. The exclusion statistics, characterized by the energy-independent parameter $\alpha \geqslant 0$, is manifested between particles within the same infinitesimal energy interval $\delta\epsilon$ [4]. The values $\alpha = 0$ and $\alpha = 1$ correspond to Bose and Fermi statistics, respectively. With this notation, the average population of a single-particle state is [4]

$$n(\epsilon) = \{w(\zeta_\epsilon) + \alpha\}^{-1} \tag{1}$$

where w satisfies the equation $w(\zeta_\epsilon)^\alpha[1 + w(\zeta_\epsilon)]^{1-\alpha} = \zeta_\epsilon^{-1} \equiv e^{(\epsilon-\mu)/k_B T}$, in obvious notation (see appendix B for a special case). The grand canonical thermodynamic potential, $\Omega \equiv -PV = k_B T \int_0^\infty d\epsilon\, C\epsilon^s \log\{(1 - \alpha n)/[1 + (1 - \alpha)n]\}$ [4], can be put in the form

$$PV = \frac{1}{s+1} \int_0^\infty d\epsilon\, C\epsilon^{s+1} n(\epsilon) \equiv \frac{U}{s+1} \tag{2}$$

where U is the internal energy. The total number of particles is $N = \int_0^\infty d\epsilon\, C\epsilon^s n(\epsilon)$. All these functions may be calculated by expressing ϵ in terms of w, but the integrals cannot be performed analytically for general s and T. Nevertheless, for $s = 0$ ($\sigma \equiv C$), all the thermodynamic quantities can be expressed in terms of elementary or polylogarithmic functions. I start with

$$N = k_B T \sigma \log(1 + y_0) \tag{3}$$

where y_0 satisfies the equation $(1 + y_0)^{1-\alpha}/y_0 = \zeta^{-1} \equiv e^{-\mu/k_B T}$. Note that y_0 depends on N and T, but not on α. Moreover, $0 = y_0(N, T = \infty) \leqslant y_0(N, T) \leqslant y_0(N, T = 0) = \infty$. From equation (3) follows

$$\exp[(\mu - \alpha N/\sigma)/k_B T] = 1 - \exp[-N/(\sigma k_B T)] \tag{4}$$

where I identify the (generalized) Fermi energy as $\epsilon_F \equiv \lim_{T\to 0} \mu = \alpha N/\sigma$ and observe that $\mu - \epsilon_F$ is also independent of α. After some algebra I get the desired expression for Ω and U:

$$\Omega = -U = (k_B T)^2 \sigma \left[\frac{1-\alpha}{2} \log^2(1 + y_0) + Li_2(-y_0) \right] \tag{5}$$

where Li_2 is Euler's dilogarithm [16]. Using the relation $Li_2(x) + Li_2[-x/(1-x)] = -\frac{1}{2}\log^2(1-x)$, valid for any $x < 1$ [16], one can prove that $\Omega \leqslant 0$ for any $\alpha \geqslant 0$ and $y_0 \geqslant 0$, as expected. Combining equations (3) and (5), it follows that

$$\Omega = -U = \frac{1-\alpha}{2}\frac{N^2}{\sigma} + (k_B T)^2 \sigma Li_2(-y_0). \tag{6}$$

In equations (3) and (6) the equivalence between ideal gases obeying *any* statistics is highlighted in the simplest way. Since y_0 does not depend on statistics, but just on N and T, it is obvious that the difference between the thermodynamic potentials of particles with different α comes just from an additive constant. All the temperature dependences are the same. As an example, one can set $\alpha = 0$ and $\alpha = 1$ and use the Landen relations [16] to re-obtain the familiar Bose and Fermi thermodynamic potentials [19, 20]. Making use of equations (6) and (3), one can obtain in the usual way expressions for the entropy and specific heat, which are both independent of α:

$$S = -k_B^2 T \sigma [2Li_2(-y_0) + \log(1+y_0)\log y_0] \tag{7}$$

$$C_V = -\frac{N^2}{T\sigma}\frac{1+y_0}{y_0} - 2k_B^2 T\sigma Li_2(-y_0). \tag{8}$$

Since, according to equation (3), $y_0 \to \infty$ when $T \to 0$, making use of the asymptotic behaviour of the dilogarithm, $Li_2(-y_0) \sim -[\pi^2/6 + \log^2(y_0)/2]$ one can show that in the limit of low temperature $C_V \sim (\pi^2/3)k_B^2 T\sigma\{1 - O[\log^2(1+y_0)/y_0]\}$. In the case of particles inside 2D boxes, the leading term is identical to the result for fermions obtained by Li *et al* [26].

2.2. Thermodynamic equivalence from the microscopic point of view: Haldane–Bose transformation

Now let me inspect this equivalence from the microscopic point of view. For this I consider a Haldane and a Bose system with the same particle number, N, and DOS, $\sigma \equiv C$. In the Haldane system I divide the energy axis into intervals of equal length, $\delta\epsilon$, and number them, starting from zero, at the lowest interval. Each of these intervals contains the same number of single-particle states, $d = \sigma\delta\epsilon$, and a variable number of particles, $n_{H,i}$ ($i \geqslant 0$). For state counting purposes I define a 'Bose dimension' of the subspace corresponding to the interval i as $d_{B,i} = d - \alpha(n_{H,i} - 1)$, so the total size of its Hilbert space is $w_{H,i} = (d_{B,i} + n_{H,i} - 1)!/[n_{H,i}!(d_{B,i} - 1)!]$ [1]. I denote $\bar{N}_{H,i} = \sum_{j=0}^{i-1} n_{H,j}$ and $\mathcal{E}_{H,i} = n_{H,i}(i\delta\epsilon - \bar{\epsilon}_{F,i})$, where $\mathcal{E}_{H,i}$ will be called the excitation energy of the particle group $n_{H,i}$, while $\bar{\epsilon}_{F,i} = \alpha\bar{N}_{H,i}/\sigma$ is the (generalized) Fermi energy of a similar system, but consisting only of $\bar{N}_{H,i}$ particles (I disregarded the distribution of particles inside the ith energy interval). Obviously, the ground-state energy of the Haldane system is $U_{H,0} = \sum_{i=0}^{\infty} n_{H,i}\bar{\epsilon}_{F,i}$ while the total energy is $U_H = \sum_{i=0}^{\infty} n_{H,i}\epsilon_{H,i} = \sum_{i=0}^{\infty} \mathcal{E}_{H,i} + U_{H,0}$. Under canonical conditions, the weight of such a distribution is $\mathcal{W}_{H,\{n_i\}} = [\prod_i \exp(-\mathcal{E}_{H,i}/k_B T)w_{H,i}]/Z_{H,N,T}$, where $Z_{H,N,T}$ is the partition function and satisfies the condition $\sum_{\{n_i\}} \mathcal{W}_{H,\{n_i\}} = 1$; the sum is taken over all distributions $\{n_i\}$. Now let me put the distributions in the Haldane system in correspondence with the distributions in the Bose system and calculate the new weights. For this, given a distribution $\{n_i\}$ in the Haldane system, I divide the energy axis of the Bose system into nonequal intervals, $\delta\epsilon_{B,i} = d_{B,i}/\sigma$, and place n_i bosons in each of them. As a consequence, the size of the Hilbert space of interval i is $w_{B,i} = w_{H,i}$. Since in a degenerate Haldane gas, in an energy interval $1/\sigma$ on average $1/\alpha$ particles coexist, to obtain the correct energy distribution one has to overlap α/σ of any consecutive energy intervals above.
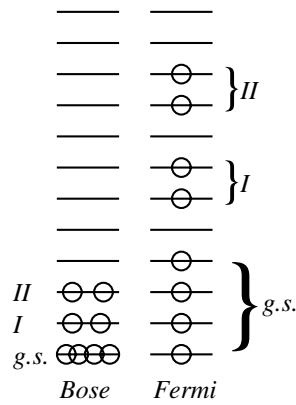
Figure 1. Excitation in Bose and Fermi gases with the same equidistant single-particle energy levels. The fermions in the g.s., I, II, ... groups are in correspondence with the bosons in the ground state, first, second, ... excited states, respectively. In this way I establish a one-to-one correspondence between the particle configurations in the Bose and Fermi systems, with the same excitation energies.

Then, the excitation energy of the group n_i is $n_i \sum_0^{i-1} \delta\epsilon_{B,j} = n_i \sum_0^{i-1} [(d - \alpha n_j)/\sigma] \equiv \mathcal{E}_{H,i}$. Therefore, the two configurations have identical statistical weights. To prove that the two systems are equivalent, I have to show that also, to any configuration in the Bose system corresponds one configuration in the Haldane system, with the same excitation energy and statistical weight. But this can be done just following the steps outlined above in reverse order. In conclusion, I showed that there is a one-to-one correspondence between the configurations in the Bose and Haldane systems. These configurations have the same 'excitation energy' and the same statistical weights, so the two systems are statistically equivalent even at the microscopic level. This analogy finds its simplest illustration in the case of ideal Bose and Fermi gases with the same spectrum, which consists of nondegenerate, equidistant single-particle energy levels (1D harmonic trap), as shown in figure 1.

From the equivalence proved above and the fact that a macroscopic Bose system with $\sigma \equiv C$ does not experience Bose condensation (see [27] for interpretations), I conclude that at any finite temperature the Fermi sea is not still (excitations occur at any depth in the Fermi sea, in macroscopic numbers). On the other hand, by transforming a Bose into a Fermi system, one may describe it (and eventually calculate interaction effects) just by considering low-energy particle–hole excitations around the Fermi surface (bosonization), which may considerably simplify the calculations.

3. Generalization: interacting systems in arbitrary traps

3.1. Statistical mechanics in boxes

I investigate here the effects of particle–particle interaction in systems inside d D containers with no external fields. The next subsection will consider arbitrary systems. In the Thomas–Fermi mean field approximation (TF-MFA), the particle–particle interaction is replaced by an effective one-body potential, $H_I(N)$ (I assume the interaction does not lead to a phase separation), and the total energy of the system is

$$E_{\alpha,\{k\}} = \sum_{\{k\}} \epsilon_k + \frac{NH_I(N\sigma^{-1})}{2} \equiv E_{ex} + U_{\alpha,g.s.}^{(0)} + \frac{NH_I(N)}{2} \qquad (9)$$

where $\epsilon_{\mathbf{k}}$ are the single-particle energies and $U_{\alpha,\text{g.s.}}^{(0)}$ is the ground-state energy of the system without interaction. The quantum numbers that specify the single-particle states are denoted by $\mathbf{k}$ (in the case of spinless particles, $\mathbf{k}$ is the momentum). If I denote the energy of the ideal system by $E_{\alpha,\{\mathbf{k}\}}^{(0)} \equiv \sum_{\{\mathbf{k}\}} \epsilon_{\mathbf{k}}$, then the excitation energy (for 2D boxes this is the energy of the equivalent Bose system, as defined in section 2.2) is $E_{\text{ex},\{\mathbf{k}\}} \equiv E_{\alpha,\{\mathbf{k}\}}^{(0)} - U_{\alpha,\text{g.s.}}^{(0)}$. In the microcanonical ensemble N is fixed, so the partition function is just a function of the excitation energy E_{ex}: $\bar{Z}_{\alpha,N}(E_{\text{ex}})$. The canonical partition function is usually written as

$$
\begin{aligned}
Z_{\alpha,N}(\beta) &= \int_{U_{\alpha,\text{g.s.}}}^{\infty} \mathrm{d}E_\alpha \, \mathrm{e}^{-\beta E_\alpha} \bar{Z}_{\alpha,N}(E_{\text{ex}}) \\
&= \mathrm{e}^{-\beta N H_{\mathrm{I}}(N)/2} \underbrace{\int_{U_{\alpha,\text{g.s.}}^{(0)}}^{\infty} \mathrm{d}E_\alpha^{(0)} \, \mathrm{e}^{(-\beta E_\alpha^{(0)})} \bar{Z}_{\alpha,N}(E_{\text{ex}})}_{Z_{\alpha,N}^{(0)}(\beta)} \\
&= \mathrm{e}^{-\beta U_{\alpha,\text{g.s.}}} \underbrace{\int_{0}^{\infty} \mathrm{d}E_{\text{ex}} \, \mathrm{e}^{-\beta E_{\text{ex}}} \bar{Z}_{\alpha,N}(E_{\text{ex}})}_{Z_{0,N}^{(0)}(\beta)}
\end{aligned}
\tag{10}
$$

where I used the shorter notation $\beta \equiv (k_{\mathrm{B}}T)^{-1}$ and $U_{\alpha,\text{g.s.}} \equiv U_{\alpha,\text{g.s.}}^{(0)} + \frac{1}{2}N H_{\mathrm{I}}(N)$. I also denoted the canonical partition function of the ideal Haldane gas of parameter α by $Z_{\alpha,N}^{(0)}(\beta)$ (or $Z_{\alpha,N}^{(0)}(T)$). $Z_{0,N}^{(0)}(\beta)$ corresponds to an equivalent Bose gas of excitations. The coefficients in front of $Z_{\alpha,N}^{(0)}(\beta)$ and $Z_{0,N}^{(0)}(\beta)$ in the two expressions for $Z_{\alpha,N}(\beta)$ are redundant in the canonical ensemble, but they should not be ignored in grand-canonical calculations.

Let me now find the population of the single-particle energy levels. Since by changing the single-particle quantum numbers $\mathbf{k}$ into $\mathbf{k}'$ the total energy of the system changes by $\epsilon_{\mathbf{k}}' - \epsilon_{\mathbf{k}}$, the relative occupation probability of the states $\mathbf{k}$ and $\mathbf{k}'$ is the same as in the case of the ideal Haldane system of parameter α. These occupation probabilities in the canonical ensemble follow from $Z_{\alpha,N}^{(0)}(\beta)$ which may be calculated by the usual saddle point method from the grand-canonical ensemble of the ideal system (after dropping the term $N H_{\mathrm{I}}(N)/2$ from the expression of total energy)—see [28] for the special case of Bose–Einstein condensed systems. In this way, I obtain the average population of the state $\mathbf{k}$, which is given by equation (1) with $\epsilon \equiv \epsilon_{\mathbf{k}}$ and the *apparent* chemical potential, μ_{a}, which in 2D satisfies equation (3). Note also that μ_{a} is not the *real* chemical potential, which is defined as

$$
\mu \equiv -k_{\mathrm{B}}T \frac{\partial \log(Z_{\alpha,N}(T))}{\partial N} = \frac{\mathrm{d}U_{\alpha,\text{g.s.}}}{\mathrm{d}N} \underbrace{- k_{\mathrm{B}}T \frac{\partial \log\left(Z_{0,N}^{(0)}(T)\right)}{\partial N}}_{\equiv \mu_{\mathrm{B}}}.
\tag{11}
$$

Taking out the ground-state energy from the total energy of the system, as done in the expressions (10), exposes the contribution to the partition function coming just from the excitations, which have a bosonic character. Then μ_{B} may be called the 'chemical potential' of these excitations. Although $\lim_{T \to 0} Z_{0,N}^{(0)}(T) = 1$, for any N (I assumed a nondegenerate ground state), implies $\lim_{T \to 0} \mu_{\mathrm{B}}(T) = 0$, unlike the chemical potential of an ideal Bose system, μ_{B} may also take positive values, as for example in the case of ideal 1D fermions. In the case of particles in a 2D box, the excitation partition function is just the partition function of the equivalent Bose system. In appendix A, I apply equations (10) to calculate the fluctuation of the ground-state population in a nonideal Bose–Einstein condensed system in a grand-canonical ensemble, while in appendix B I discuss the MFA' model in more detail.

3.2. General trapping potential: thermodynamic considerations

To apply the procedure introduced in section 3.1 from a macroscopic point of view, I shall first show some general properties of the entropy function of a neutral fluid, characterized by the extensive parameters U (internal energy), V (volume) and N (particle number): $S(U, V, N)$. Non-essential parts of the proof will be skipped. In what follows, the parameters omitted in the expressions are the parameters held fixed. S is a positive, homogeneous function of order one, concave downwards [29] $[S(\lambda U, \lambda V, \lambda N) = \lambda S(U, V, N)$ and $S(U_1 + U_2, V_1 + V_2, N_1 + N_2) \geqslant S(U_1, V_1, N_1) + S(U_2, V_2, N_2)]$. The inverse function $U(S, V, N)$ is concave upwards. If I write again $U(S, V, N) = U_{\text{g.s.}}(V, N) + U_{\text{B}}(S, V, N)$, then $U_{\text{g.s.}}(V, N) \equiv U(S = 0, V, N)$ is also concave upwards (but, in the general case, U_{B} may not have this property). I take $U_{\text{g.s.}}(V, N = 0) = 0$. Now I fix V. I assume that N and $U(S, N_{\text{fixed}})$ take values in the intervals $[0, \infty)$ and $[U_{\text{g.s.}}(N), \infty)$, respectively (the limits may be restricted further, such as for example in systems of spins or hard-core particles, but here I work with these intervals). The concavity property and the range of U implies that $S(U)$ is monotonic (increasing). If I fix $U \equiv U_{\text{fixed}}$ and $U_{\text{g.s.}} \equiv 0$, then $S(U_{\text{fixed}}, N)$ is also monotonic in N, otherwise it is zero at both ends of the allowed interval along the N axis. If I introduce the function S_{B} by $S_{\text{B}}(U_{\text{B}}, V, N) = S(U, V, N)$, then S_{B} and S offer two equivalent descriptions of the system. Moreover, if S_{B} is concave downwards and coincides with the entropy of a new system, say B, then we may say that our original system, A, is *thermodynamically equivalent* with B (I denote it by A$\sim$ B). In such a case, if $U_{\text{Cg.s.}}(V, N)$ is any homogeneous function of order one, concave upwards, then the entropy $S_{\text{C}}(U_{\text{Cg.s.}} + U_{\text{B}}, V, N) = S_{\text{B}}(U_{\text{B}}, V, N)$ is a legitimate entropy (with all required properties). Moreover, if $S_{\text{C}}(U_{\text{Cg.s.}} + U_{\text{B}}, V, N)$ describes a system C, then A $\sim$ B $\sim$ C. Therefore for any system described by $S(U, V, N)$ I can define the *Bose entropy S_{B}* by the procedure outlined above. According to the definition given in section 1, the systems with the same S_{B} are called *thermodynamically equivalent* and they form an equivalence class.

Since the number of microstates available in a system is dependent on the excitation energy E_{ex}, V and N ($U_{\text{g.s.}}(V, N)$ is a redundant quantity), one can calculate partition functions having as their starting point the Bose entropy of the reservoir, $\mathcal{S}_{\text{B}}$. In the grand-canonical ensemble, the probability associated with a microstate $w(E_{\text{ex}}, V, N)$ of the system is proportional to $\exp\left[k_{\text{B}}^{-1}\mathcal{S}_{\text{B}}(\mathcal{E}_{\text{ex}}, \mathcal{V}, \mathcal{N})\right]$ (I use calligraphic letters for the reservoir quantities). Writing as usual the Taylor expansion of the reservoir entropy and identifying $U_{\text{B}} \equiv E_{\text{ex}}$, I obtain the probability of the microstate w:

$$p(w) = \frac{e^{-\beta(U_{\text{B}} - \mu_{\text{B}}N)}}{\mathcal{Z}} \tag{12}$$

where $\mathcal{Z}$ is the partition function, while μ_{B} is the chemical potential of the bosonic excitations, defined by equation (11).

Obviously, the 2D Haldane gases described in the MFA form an equivalence class. Their Bose entropy is the entropy of the equivalent ideal Bose gas. Since μ_{B} is the same for all systems in an equivalence class, the difference between the chemical potentials of different systems is due only to the ground-state energy (equation (11)). Whether the statistics, the interaction, or both, are responsible for the dependence of the ground-state energy on N, is of no importance. Two equivalent gases of different statistics (I also showed in section 3.1 that the MFA does not change the microscopic exclusion statistics properties), but with the same $U_{\text{g.s.}}(N, V)$, have identical behaviour even in general trapping potentials, as long as the Thomas–Fermi approximation holds (so that a system in a variable trapping potential can be regarded as a collection of boxes with imaginary walls, in contact with each other).

4. Conclusions

In summary, in the beginning of the paper I gave simple expressions, in terms of polylogarithmic functions, for the thermodynamic quantities characterizing Haldane ideal systems with constant density of single-particle states. These expressions are easy to handle and show in a most simple way the thermodynamic equivalence (the same entropy function) of such systems. Second, I showed the microscopic reason for this equivalence, which is the similarity between the excitation spectra. The model used for this purpose is different from the usual bosonization technique (see [30] for a review) and enables one to transform any Haldane system into a Bose system, and vice versa. These transformations: Bose↔Haldane↔Fermi, might be very useful if, for a given interaction, techniques developed for one kind of system (such as bosonization for Fermi systems) can be transferred to other types of system (at low temperatures, even in Bose systems, the effective thermodynamic contribution comes from a small subset of single-particle states).

In section 3, I gave a general interpretation of this correspondence technique, by subtracting from the internal energy of the system, U, the ground-state energy, $U_{\text{g.s.}}$. This led to an equivalent writing of the partition function, in which the internal energy is replaced by the excitation energy, $U - U_{\text{g.s.}}$. In this formulation the chemical potential should be redefined (equation (11)). Moreover, this redefinition of entropy provides a clearer definition of the thermodynamic equivalence, which splits the set of physical systems into equivalence classes. Systems belonging to the same class have similar excitation spectra (at least up to corrections that vanish in the thermodynamic limit). In appendix A, I apply the results of section 3 to calculate particle fluctuations in a non-ideal, condensed Bose system.

In appendix B, I show some interesting characteristics of what I called the MFA$'$ model and discuss similarities and differences with the mean field approximation and fractional exclusion statistics. Although on microscopic intervals along the pseudoenergy axis, the MFA$'$ model reproduces the characteristics of FES gases, as defined by Haldane [1] and Wu [4], in some cases a rigorous calculation of the population of single-particle states reveals a condensation phenomenon which also implies a malfunction of the usual calculation technique.

Acknowledgments

Appendix A. Particle fluctuation in interacting Bose–Einstein gases

I apply here the technique described in section 3 to calculate the fluctuation of the ground state occupation number, N_0, of a condensed nonideal Bose gas, in the grand-canonical ensemble. The interaction is assumed to produce a particle-dependent ground-state energy. Let me denote by $\mathcal{Z}(\beta, \beta\mu)$ the grandcanonical partition function. Using the notation and arguments from section 3, I write

$$\mathcal{Z}(\beta, \beta\mu) = \sum_N \mathrm{e}^{-\beta(U_{\text{g.s.}} - \mu N)} \underbrace{\int_0^\infty \mathrm{d}E_{\text{ex}}\, \mathrm{e}^{-\beta E_{\text{ex}}} \bar{Z}_N(E_{\text{ex}})}_{Z_N^{(0)}(\beta)} \equiv \sum_N \mathcal{Z}_N(\beta, \beta\mu)$$

and I shall omit the subscripts denoting $\alpha = 0$. $Z_N^{(0)}(\beta)$ is the canonical partition function of the ideal Bose system. As shown in section 3.2, $U_{\text{g.s.}}(N)$ is a function concave upwards.

The maximum of the probability distribution over the particle number is given by the equation $\partial \mathcal{Z}_N(\beta, \beta\mu)/\partial N = 0$, where

$$\frac{\partial \mathcal{Z}_N(\beta, \beta\mu)}{\partial N} = \mathcal{Z}_N(\beta, \beta\mu) \left[-\beta \frac{\mathrm{d}U_{\mathrm{g.s.}}}{\mathrm{d}N} + \beta\mu - \beta\mu_B \right] \equiv \mathcal{Z}_N(\beta, \beta\mu) \Psi_N(\beta, \beta\mu).$$

The solution of the equation will be denoted by $N_{\max}$. If there is no $N_{\mathrm{bound}} < \infty$, so that $\lim_{N \to N_{\mathrm{bound}}}[\mathrm{d}U_{\mathrm{g.s.}}/\mathrm{d}N] = \infty$, $N_{\max}$ increases to infinity as μ increases. If the system is not pathologic, then $\lim_{N \to \infty} \mu_B(N) = 0$ and for large enough μ, $N_{\max}$ is the solution of $\mathrm{d}U_{\mathrm{g.s.}}/\mathrm{d}N = \mu$. In such a situation we say that the system is Bose–Einstein condensed, since for the relevant values of E_{ex} the configuration space does not increase with N (the ground state is already a particle reservoir). The probability distribution $\mathcal{Z}_N$ may be expanded around $N = N_{\max}$ to get $Z_N(\beta, \beta\mu) \approx Z_{N_{\max}}(\beta, \beta\mu) \exp[(\partial \Psi_N/\partial N)_{N=N_{\max}}(N - N_{\max})^2/2]$ where

$$\left. \frac{\partial \Psi_N(\beta, \beta\mu)}{\partial N} \right|_{N=N_{\max}} = -\beta \frac{\mathrm{d}^2 U_{\mathrm{g.s.}}}{\mathrm{d}N^2}.$$

For fixed N, the fluctuation of N_0, $\langle \delta^2 N_0 \rangle_N$, is equal to the fluctuation of the particle number on the excited energy levels, $\langle \delta^2 N_{\mathrm{ex}} \rangle_N$ (see, for example, [31–33] for fluctuations in ideal gases). If the probability distribution of N_0 in the canonical ensemble is $P(N_0, N) = (2\pi \langle \delta^2 N_0 \rangle_N)^{-1/2} \exp[-(N_0 - \langle N_0 \rangle_N)^2/2\langle \delta^2 N_0 \rangle_N]$, then the grand-canonical probability distribution of N_0 is

$$\mathcal{P}(\beta, \beta\mu, N_0) = \sqrt{\frac{\beta}{2\pi} \frac{\mathrm{d}^2 U_{\mathrm{g.s.}}}{\mathrm{d}N^2}} P(N_0, N) \exp\left[-\frac{\beta}{2} \frac{\mathrm{d}^2 U_{\mathrm{g.s.}}}{\mathrm{d}N^2} (N - N_{\max})^2 \right]. \tag{A.1}$$

Since the system is condensed, the *grand-canonical* fluctuation of the total particle number $\langle \delta^2 N \rangle$ is equal to the fluctuation of the *canonical average* ground state occupation number. Using this observation and equation (A.1), the total fluctuation of N_0 in the grand-canonical ensemble follows:

$$\begin{aligned}
\langle \delta^2 N_0 \rangle_{\beta, \beta\mu} &= \int \mathrm{d}N \int \mathrm{d}N_0 \left(N_0 - \langle N_0 \rangle_{N_{\max}} \right)^2 \mathcal{P}(\beta, \beta\mu, N_0) \\
&= \langle \delta^2 N_0 \rangle_{N_{\max}} + \langle \delta^2 N \rangle_{\beta, \beta\mu}.
\end{aligned}$$

Therefore, the grand-canonical total particle fluctuation of the nonideal gas adds quadratically to the canonical fluctuation of the ground-state population, to give the total mean square fluctuation of the number of particles in the condensate.

Appendix B. The MFA′ model versus mean field approximation and fractional exclusion statistics

If the interaction Hamiltonian of section 3 is $H_I(N) \equiv \sigma^{-1} g N$, where g is a dimensionless constant and $\sigma \equiv C$ (constant), then (mostly) repeating the arguments from [11, 12, 13, 15] I define the MFA′ model by assimilating the interaction energy from equation (9) into the single-particle (kinetic) energies $\epsilon_{\mathbf{k}}$, and I define the pseudoenergies

$$\tilde{\epsilon}(\epsilon_{\mathbf{k}}) \equiv \epsilon_{\mathbf{k}} + \sum_{\mathbf{k}'} \frac{g}{\sigma} h(\tilde{\epsilon}(\epsilon_{\mathbf{k}}) - \tilde{\epsilon}(\epsilon_{\mathbf{k}'})) n(\mathbf{k}') \tag{B.1}$$

where $n(\mathbf{k}')$ is the population of the state $\mathbf{k}'$ and $h(x) = 1, h_0$, or 0, depending on whether $x > 0, x = 0$, or $x < 0$, respectively; in what follows I assume $g \geqslant -\alpha$ and $h_0 \in [0, 1]$, where α is the statistics parameter of a gas without interaction[1]. As Wu [4], I divide the

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[1] In the counting rule of Hansson *et al* [12], two particles could not have the same pseudoenergy even if they occupy the same single-particle state. Since I work with a (quasi)continuous spectrum, I adopt the more general point of view of Isakov and Viefers [13], but relaxing also the constraint $h_0 \equiv \frac{1}{2}$.

energy and pseudoenergy axes into infinitesimal (microscopic) intervals. If such microscopic divisions exist for which none of the intervals is completely blocked, then the energy interval $[\epsilon_{\mathbf{k}}, \epsilon_{\mathbf{k}} + d\epsilon)$ is transformed into the pseudoenergy interval $[\tilde{\epsilon}_{\mathbf{k}}, \tilde{\epsilon}_{\mathbf{k}} + d\tilde{\epsilon})$, where $d\tilde{\epsilon} = d\epsilon + \sigma^{-1} g\rho(\epsilon_{\mathbf{k}}) \, d\epsilon = d\epsilon + \sigma^{-1} g\tilde{\rho}(\tilde{\epsilon}) \, d\tilde{\epsilon}$, and $\sigma n(\epsilon_{\mathbf{k}}) \, d\epsilon \equiv \rho(\epsilon_{\mathbf{k}}) \, d\epsilon = \tilde{\rho}(\tilde{\epsilon}) \, d\tilde{\epsilon}$ represent the average number of particles in the intervals under consideration. Moreover, the Bose dimensions of the two intervals $d\tilde{\epsilon}$ and $d\epsilon$ are

$$d_{\mathrm{B,d}\epsilon} = \sigma \, d\epsilon - \alpha\rho(\epsilon_{\mathbf{k}}) \, d\epsilon = \sigma \, d\tilde{\epsilon} - (\alpha + g)\tilde{\rho}(\tilde{\epsilon}) \, d\tilde{\epsilon} \equiv d_{\mathrm{B,d}\tilde{\epsilon}} \tag{B.2}$$

where $\sigma - g\tilde{\rho}(\tilde{\epsilon})$ is the *effective* density of states along the pseudoenergy axis. The Bose density $d_{\mathrm{B,d}\tilde{\epsilon}}$ is identical to the Bose density of an ideal Haldane gas of parameter $\alpha + g$ (see also [34] for related discussions). This implies, in accordance with [11, 13], that the MFA$'$ model leads to the same microscopic particle distribution $\tilde{\rho}(\tilde{\epsilon})$ as an ideal Haldane gas of statistics parameter $\alpha + g$, call it $n_{\alpha+g}(\tilde{\epsilon})$ (1) (I included the subscript $\alpha + g$ in the notation to specify the statistics parameter of an ideal Haldane gas).

Appendix B.1. 2D Bose gas in the MFA$'$ approximation: an unexpected condensation

To continue exercising with the MFA$'$ approximation, let me now consider a 2D interacting Bose gas. I assume that the particle–particle interaction is repulsive, so $H_{\mathrm{I}}(N) \equiv \sigma^{-1} gN$, with $g > 0$. Under the canonical condition and in the MFA approximation, the gas is thermodynamically equivalent to the ideal Bose gas and its entropy and specific heat are given by the universal expressions (7) and (8). On the other hand, in the MFA$'$ model the energies of the single-particle states are 'shifted' according to equation (B.1), but the occupation of the single-particle states is done in accordance with the 'original' Bose statistics (see equation (B.2)). If I number the single-particle (kinetic) energies and pseudoenergies as $\epsilon_0 = 0 < \epsilon_1 \leqslant \cdots$, and $\tilde{\epsilon}_0 < \tilde{\epsilon}_1 \leqslant \cdots$, respectively (I assume that level 0 is nondegenerate), then the MFA$'$ grand-canonical partition function is

$$\tilde{\mathcal{Z}}(\beta, \beta\mu) \equiv \sum_{\{n_i\}} \mathrm{e}^{-\beta\sigma^{-1} gh_0 n_0^2 + \beta\mu n_0} \prod_{i \geqslant 1} \mathrm{e}^{-\beta\tilde{\epsilon}_i n_i + \beta\mu n_i}$$

$$\equiv \sum_{n_0} \mathrm{e}^{-\beta\sigma^{-1} gh_0 n_0^2 + \beta\mu n_0} \tilde{\mathcal{Z}}_{\mathrm{ex}}^{\{n_i\}} \tag{B.3}$$

where $\tilde{\mathcal{Z}}_{\mathrm{ex}}^{\{n_i\}} \equiv \sum_{\{n_i\}} \prod_{i \geqslant 1} \mathrm{e}^{-\beta\tilde{\epsilon}_i n_i + \beta\mu n_i}$, while n_i represents the population of the single-particle state i. The exponent $-\beta\sigma^{-1} gh_0 n_0^2 + \beta\mu n_0$ has a maximum for $n_0 \equiv n_{0,\max} = \sigma\mu(2gh_0)^{-1}$. If $\mu < 0$, $n_{0,\max} < 0$, and I expect no macroscopic population of any single-particle state. In such a case, according to equation (B.2), $\tilde{\rho}(\tilde{\epsilon}) = \sigma n_g(\tilde{\epsilon})$ and I should recover the previous results for the ideal Haldane system of parameter g (see [4]). This will be shown to be true and for the Haldane gas of parameter $g > 0$ (with the chemical potential μ_{H}) under canonical conditions, a strictly positive temperature T_{inv} exists (defined by equation (4)), so that $\mu_g(T < T_{\mathrm{inv}}) > 0$. Since for $T > T_{\mathrm{inv}}$, $\mu = \mu_{\mathrm{H}}$, I conclude that $\mu(T > T_{\mathrm{inv}}) \to 0$ as $T \to T_{\mathrm{inv}}$. Turning back to the MFA$'$ model in the grand-canonical ensemble, I investigate the situation $\mu \geqslant 0$. If $\mu > 0$, the distribution $\mathrm{e}^{-\beta\sigma^{-1} gh_0 n_0^2 + \beta\mu n_0}$ has a very sharp maximum centred at $n_0 = n_{0,\max} > 0$ and with the second moment $\delta^2 n_0 = (\beta\mu)^{-1} n_{0,\max}$. The relative fluctuation of n_0 is $\sqrt{\delta^2 n_0}/n_{0,\max} = (\beta\mu)^{-1/2} n_{0,\max}^{-1/2}$, which is zero in the thermodynamic limit. Most often, such a sharp distribution is believed to fix the average value of the variable at the distribution maximum, but this is an example where this general procedure would lead to wrong results.

For a rigorous calculation, I denote the grand-canonical partition function of the Haldane system of parameter g by $\mathcal{Z}_g(\beta, \beta\mu)$ and assume as a beginning that $n_{i \geqslant 1}$ are all microscopic

populations (at the end this will turn out to be true). Then

$$\tilde{\mathcal{Z}}(\beta, \beta\mu) \equiv \sum_{n_0} \tilde{\mathcal{Z}}_{n_0}(\beta, \beta\mu) = \sum_{n_0} e^{-\beta\sigma^{-1}gh_0 n_0^2 + \beta\mu n_0} \mathcal{Z}_g[\beta, \beta(\mu - \tilde{\epsilon}_1)]$$

$$\approx \sum_{n_0} e^{-\beta\sigma^{-1}gh_0 n_0^2 + \beta\mu n_0} \mathcal{Z}_g \left[\beta, \beta\left(\mu - \frac{gn_0}{\sigma}\right)\right]. \tag{B.4}$$

The most probable value of n_0 in the summation (B.4) is given by the equation $\partial \tilde{\mathcal{Z}}_{n_0}/\partial n_0 = 0$. To find this value I define the function

$$f_{h_0}(n_0, \mu) \equiv \tilde{\mathcal{Z}}_{n_0}^{-1} \frac{\partial \tilde{\mathcal{Z}}_{n_0}}{\partial n_0} = -2h_0 n_0 \frac{\beta g}{\sigma} + \beta\mu - \langle N_{\text{ex}} \rangle \frac{\beta g}{\sigma} \tag{B.5}$$

where $\langle N_{\text{ex}} \rangle \equiv \partial \log(\mathcal{Z}_g)/\partial(\beta\mu')$ is the average number of particles on the excited energy levels. The most probable value of n_0 is given by one of the zeros of f. For simplicity I shall use the notations $\xi \equiv \beta n_0 g/\sigma$, $\zeta \equiv \beta\mu$ and $\mu' \equiv \mu - \sigma^{-1}gn_0$. Plugging equation (3) into (B.5) I can rewrite f in two equivalent ways:

$$f_{h_0}(\xi, \zeta) = \log\left(\frac{y_0'}{1 + y_0'}\right) + (1 - 2h_0)\xi \tag{B.6}$$

$$= (1 - 2h_0)\zeta + 2h_0 \log(y_0') - [2h_0 + g(1 - 2h_0)] \log(1 + y_0') \tag{B.7}$$

where $(1 + y_0')^{1-g}/y_0' = e^{-(\zeta-\xi)}$. Expression (B.6) implies that $f_{h_0}(0, \zeta) < 0$ for any h_0 and $f_{h_0}(\xi, \zeta) < 0$ for any ξ, if $h_0 \geq 1/2$. Therefore the probability distribution of n_0 has a local maximum at $n_0 = 0$. If $h_0 \geq 1/2$, then $n_0 = 0$ is the only maximum of f_{h_0}. Since $\tilde{\mathcal{Z}}_{n_0} = \mathcal{Z}_g(\beta, \beta\mu) \exp\left(g^{-1}k_B T\sigma \int_0^{\xi(n_0)} f_{h_0}(\xi', \zeta) \, d\xi'\right)$ and $k_B T\sigma$ is assumed to be a macroscopic quantity, it follows that $\tilde{\mathcal{Z}}_{n_0}$ has an infinitely sharp maximum at $n_0 = 0$ which implies that the ground-state is microscopically populated in spite of the very sharp maximum of $e^{-\beta\sigma^{-1}gh_0 n_0^2 + \beta\mu n_0}$, centred at $n_{0,\text{max}}$.

To observe the behaviour of f_{h_0} for large ξ we have to look at the expression (B.7). For $h_0 = 0$, the function has the simple form $f_0(\xi, \zeta) = \zeta - g \log(1 + y_0')$. Since $y_0'(\zeta - \xi)$ decreases monotonically from $y_0'(\zeta)$ to $y_0'(-\infty) = 0$, as ξ increases from 0 to ∞, then $f_0(\xi, \zeta)$ is also a monotonic function of ξ, with $f_0(0, \zeta) < 0$ and $f_0(\infty, \zeta) = \zeta > 0$. The only zero of f_0 is at $y_0' = y_{0,\text{min}}' \equiv e^{\zeta/g} - 1$ and corresponds to a minimum of probability. As one can observe directly from equation (B.3), the maximum probability is ∞ and corresponds to $n_0 = \infty$. Yet, as I mentioned above, the system has a metastable state for microscopic n_0, which corresponds to the equivalent (ideal) Haldane distribution.

The function $f_{h_0}(\xi, \zeta)$ is a continuous function in all the parameters and variables. From equation (B.6) it follows that for fixed ξ and ζ, f is a decreasing function of h_0 and from equation (B.7) we observe that for $\xi \gg \zeta$, $f_{h_0}(\xi, \zeta) \approx \zeta - 2h_0\xi$. For small enough h_0, $f_{h_0}(\xi, \zeta) = 0$ has a solution at $\xi_{\text{max}} \approx \zeta/(2h_0)$ and $\exp(\zeta - \xi_{\text{max}}) = \exp\{-[(2h_0)^{-1} - 1]\zeta\} \ll \exp(\zeta/g) - 1$. In such a case, and since $f_{h_0>0}(\xi \to \infty, \zeta) \to -\infty$ and also $f_{h_0}(0, \zeta) < 0$, I can conclude that f has two zeros. The first corresponds to a local minimum of probability for n_0, while the second corresponds to a local maximum. Whether $n_0 = 0$ or $n_0 = n_{0,\text{max}} \equiv \xi_{\text{max}}\sigma/(\beta g) \approx \sigma\mu(2gh_0)^{-1}$ has higher probability, depends on the specific values of h_0 and ζ. Nevertheless, continuity of f and monotonicity in h_0 imply that a critical value of h_0 exists between 0 and 1/2, and which increases monotonically with ζ, I call it $h_{0,\text{cr}}(\beta\mu)$, so that for $h_0 < h_{0,\text{cr}}(\beta\mu)$ the ground state is macroscopically populated (the probability for $n_0 = n_{0,\text{max}}$ is highest), while for $h_0 > h_{0,\text{cr}}$ the ground state has microscopic occupation number and is well described by the Haldane ideal gas of parameter g. If the ground state is macroscopically populated, I say that *the 2D MFA′ Bose system is condensed*.

Now I prove that none of the excited energy levels is macroscopically occupied. Obviously, I start with $\tilde{\epsilon}_1$. This level can be macroscopically occupied *if and only if* $\mu'(n_{0,\max}) \geqslant 0$ and $h_0 \leqslant h_{0,\text{cr}}(\beta\mu')$. The condition $\mu' \geqslant 0$ implies $h_0 \leqslant h_{0,\text{cr}}(\beta\mu)$ and $n_{0,\max} \leqslant g^{-1}\sigma\mu$. But for $n_0 = g^{-1}\sigma\mu$, $\xi = \zeta$ and $f_{h_0}(\zeta, \zeta) = \log[y_0(0)] - \log[1 + y_0(0)] + (1 - 2h_0)\zeta$—where $y_0(0) > 1$ is a fixed value which satisfies the equation $y_0(0)(1 + y_0(0))^{g-1} = 1$. Therefore $f_{h_0}(\zeta, \zeta)$ is a function linear in ζ, which starts at $f_{h_0}(0, 0) = \log[y_0(0)] - \log[1 + y_0(0)] < 0$ and, since $h_0 \leqslant h_{0,\text{cr}}(\beta\mu) < 1/2$, increases to infinity as ζ increases. The continuity of $f_{h_0}(\xi, \zeta)$ then completes the proof that $\zeta < \xi_{\max}$ always, which implies that $n_{0,\max} > g^{-1}\sigma\mu$. This proves the fact that $\tilde{\epsilon}_1$ and as a consequence any state of pseudoenergy $\tilde{\epsilon}_{i \geqslant 1}$ is microscopically populated. Also, the monotonic increase of $h_{0,\text{cr}}$ with ζ implies that, if the ground state is microscopically populated, so are all the other states.

To finish the exercise I will show that for any $0 \leqslant h_0 < 1/2$, a temperature $T_c < T_{\text{inv}}$ exists at which the system condenses on the ground state. If the system is not condensed, it is described as a Haldane gas of parameter g. In this case the relative fluctuation of the total particle number N vanishes in the thermodynamic limit. On the other hand, for a condensed system,

$$
\begin{aligned}
\tilde{\mathcal{Z}}_{n_0} &= \tilde{\mathcal{Z}}_{n_{0,\max}} \exp\left(\frac{k_B T \sigma}{g} \int_{\xi_{\max}}^{\xi(n_0)} f_{h_0}(\xi', \zeta)\, \mathrm{d}\xi' \right) \\
&\approx \tilde{\mathcal{Z}}_{n_{0,\max}} \exp\left(\frac{\beta g}{2\sigma}(n_0 - n_{0,\max})^2 \left. \frac{\partial f_{h_0}(\xi, \zeta)}{\partial \xi} \right|_{\xi = \xi_{\max}} \right)
\end{aligned}
\tag{B.8}
$$

where I used the fact that $f_{h_0}(\xi_{\max}, \zeta) = 0$. From (B.8) it is easy to observe that the relative fluctuation is proportional to $n_{0,\max}^{-1/2}$, which vanishes in the thermodynamic limit. The relative total particle fluctuation, which is obtained by adding together the contributions from the ground state, from the excited states (described as a Haldane gas), and the correlations between them, vanishes also. It is therefore safe to use the grand-canonical average values in canonical calculations even for this unusual toy system (fine-tuning due to the change of ensemble is only relevant for the finite size effects). At $T = T_{\text{inv}}$, $\mu = 0$ and so is ζ. In this case, from equation (B.7) we see that $f_{h_0}(\xi, 0) < 0$ for any ξ and h_0, therefore n_0 is still microscopic, so the condensation temperature is lower. If the system does not condense, μ increases as the temperature is lowered, which implies an even faster increase of ζ. If I assume that for a chosen $h_0 < 1/2$, $T_c = 0$ (the system does not condense), then I can choose T so that ζ takes any value up to infinity. If $\zeta - \xi \gg 1$, then $y_0' \approx e^{(\zeta-\xi)/g}$ and $\log[(1 + y_0')/y_0'] \approx e^{-(\zeta-\xi)/g} \ll 1$. Now, for any $h_0 < 1/2$ and interval $[0, \xi_0]$, I can choose $\zeta \gg 1 + \xi_0$, so that $f_{h_0}(\xi_0, \zeta) > 0$ and $\int_0^{\xi_0} \log[(1 + y_0')/y_0'] < (1 - 2h_0)\xi_0^2/2$, which implies that $\tilde{\mathcal{Z}}_{n_{0,\max}} > \tilde{\mathcal{Z}}_{n_0(\xi_0)} > \tilde{\mathcal{Z}}_0 = \mathcal{Z}_g(\beta, \beta\mu)$. In this case the system is condensed, with $n_{0,\max}$ particles on the ground state, so the initial assumption ($T_c = 0$) was wrong.

In conclusion, for any $h_0 < 1/2$, a temperature $T_c \in (0, T_{\text{inv}})$ exists at which the condensation occurs and below which the similarity between the MFA$'$ Bose system and the 'usual' ideal Haldane system [4] of parameter g is lost. Moreover, the onset of condensation also removes any thermodynamic equivalence between gases described in the MFA and MFA$'$ models. The exercise presented here is also interesting due to the fact that it showed in a concrete example how an infinitely sharp (in the thermodynamic sense) probability distribution of particle on the ground state may be overwhelmed by the probability distribution of particles on the excited energy levels. Vice versa, it also shows that the equilibrium distribution of the Haldane gas, as deduced by Isakov [2, 3] and Wu [4], and which corresponds to microscopic n_0, may not be the equilibrium distribution, in spite of the FES manifested in equation (B.2).

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